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LETTER TO THE EDITOR

Generalized symmetries and algebras of the two-dimensional differential-difference Toda equation

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Abstract. Applying the formal series symmetry theory to a differential-difference system, two-dimensional Toda equation, we found that as with the continuous Toda field theory, there exist two sets of infinitely many symmetries for the discrete Toda system. Each set of symmetries constitutes a generalized W_∞ algebra.

The study of the W_∞ has attracted much attention from physicists and mathematicians because its wide applications in various physics fields such as string theory [1], conformal field [2], two-dimensional gravity [3], membrane theory [4] and integral models [5], have been found. Recently, one of the present authors (Lou) found that there exists much more generalized symmetry algebras for (2 + 1)-dimensional pure differential integrable models [6–10]. In this short letter, we would like to extend the method of [6, 7] to the differential-difference system. We take the well known two-dimensional discrete Toda equation (DTE) [11]

$$H \equiv u_{xt}(n) - K(u(n)) \equiv u_{xt}(n) - \mathrm{e}^{u(n-1)-u(n)} + \mathrm{e}^{u(n)-u(n+1)} = 0$$
$$(n = 0, \pm 1, \pm 2, \ldots, \pm\infty) \tag{1}$$

where field $u(n)$ is a function of the continuous variables x, t and the discrete variable n, as a typical example.

A symmetry of the DTE (1) is defined as a solution of the linearized equation of (1),

$$\sigma_{xt}(n) = K'\sigma(n) \equiv \frac{\partial}{\partial\epsilon}K(u(n)+\epsilon\sigma(n))\Big|_{\epsilon=0,H=0} \tag{2}$$
$$= (\mathrm{e}^{u(n-1)-u(n)}\Delta(n) - \mathrm{e}^{u(n)-u(n+1)}\Delta(n+1))\sigma(n)|_{H=0}$$
$$(n = 0, \pm 1, \pm 2, \ldots, \pm\infty) \tag{3}$$

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that is to say (1) is form invariant under the transformation

$$u(n) \to u(n) + \epsilon\sigma(n) \qquad (n = 0, \pm 1, \pm 2, \ldots, \pm\infty) \tag{4}$$

with infinitesimal parameter ϵ. The operator $\Delta(n)$ in (3) is defined as

$$\Delta(n)F(n) = F(n-1) - F(n) \tag{5}$$

where $F(n)$ is an arbitrary function of n.

There may be various types of symmetries of (1). However, similar to the continuous cases [6–10], in this letter we focus our interest only on the solutions of (3) which have the form:

$$\sigma(n, f) = \sum_{k=0}^{\infty} f^{(-k)}\sigma[n, k] \tag{6}$$

where $f = f(t)$ is an arbitrary function of t, $f^{(-k)} = \partial^{-k} f$ and $\partial_t^{-1} = \int^t \mathrm{d}t$.

Substituting (6) into (2) (or (3)) we have

$$\sum_{k=0}^{\infty} f^{(-k+1)}\sigma_x[n, k] = \sum_{k=1}^{\infty} f^{(-k+1)}(K' - \partial_x\partial_t)\sigma[n, k-1]. \tag{7}$$

Because f is an arbitrary function of t, (7) should be true at any order of k, that means

$$\sigma_x[n, 0] = 0 \qquad \text{i.e. } \sigma[n, 0] = g(t, n) \equiv g \tag{8}$$

$$\begin{aligned} \sigma[n, k] &= (-\partial_t + \partial_x^{-1} K')\sigma[n, k-1] \\ &= (-\partial_t + \partial_x^{-1} K')^2\sigma[n, k-2] \\ &= \cdots \\ &= (-\partial_t + \partial_x^{-1} K')^k\sigma[n, 0] \\ &= (-\partial_t + \partial_x^{-1} K')^k g \end{aligned} \tag{9}$$

where $g = g(t, n)$ is an arbitrary function of n and t while $\partial_x^{-1} = \int^x \mathrm{d}x$ is an indefinite integral operator. From (8) and (9) we get a general formal series symmetry of the discrete Toda equation (1)

$$\sigma(n, f) = \sum_{k=0}^{\infty} f^{(-k)}(\partial_x^{-1} K' - \partial_t)^k g. \tag{10}$$

For general function g, (10) is a series symmetry of (1). Similar to the pure differential systems [6–10], it is possible that the series symmetry (10) reduces to the truncated one if we select the function g properly. Fortunately, the result is quite simple.

If we fix function g simply as

$$g \equiv g_i(n) = \frac{n^{i-1}}{(i-1)!} \qquad i = 1, 2, 3, \ldots, +\infty \tag{11}$$

the similar discussion, as in [6, 7], shows us that the symmetry (10) will reduce to a truncated one:

$$\sigma \equiv \sigma_i^t(n, f(t)) = \frac{1}{(i-1)!}\sum_{k=0}^{i-1} f^{(i-k-1)}(-\partial_t + \partial_x^{-1}\mathrm{e}^{u(n-1)-u(n)}\Delta(n) - \partial_x^{-1}\mathrm{e}^{u(n)-u(n+1)}\Delta(n+1))^k n^{i-1} \qquad i = 1, 2, 3, \ldots \tag{12}$$

where the arbitrary function f has been rewritten as $f^{(i-1)}$ for convenience later.

The first four of (12) reads

$$\sigma_1^t(n, f(t)) = f(t) \tag{13}$$

$$\sigma_2^t(n, f(t)) = -f(t)v_t + f'(t)n \tag{14}$$

$$\sigma_3^t(n, f(t)) = f(t)[\partial_x^{-1}\tau_x(n-1)\tau(n-2) + \partial_x^{-1}\tau_x(n)\tau(n+1) - \tau(n)\tau(n-1)] + f'(t)[(\tfrac{1}{2}-n)\tau(n-1) + (\tfrac{1}{2}+n)\tau(n)] + \tfrac{1}{2}f'(t)n^2 \tag{15}$$

$$\begin{aligned}\sigma_4^t(n, f(t)) = {}& f(t)[\partial_x^{-1}\tau_x(n)\partial_x^{-1}\tau_x(n+1)\tau(n+2) - \partial_x^{-1}\tau_x(n-1)\partial_x^{-1}\tau_x(n-2)\tau(n-3) \\ &+ \tau(n)\partial_x^{-1}\tau_x(n-1)\tau(n-2) - \tau(n-1)\partial_x^{-1}\tau_x(n)\tau(n+1)] \\ &+ f'(t)[(n-1)\partial_x^{-1}\tau_x(n-1)\tau(n-2) + (n+1)\partial_x^{-1}\tau_x(n)\tau(n+1) - n\tau(n)\tau(n-1)] \\ &+ \tfrac{1}{6}f''(t)[(3n^2+3n+1)\tau(n) - (3n^2-3n+1)\tau(n-1)] + \tfrac{1}{6}f'''(t)n^3\end{aligned} \tag{16}$$

where primes are differentiations with respect to time t and

$$\tau(n) = \partial_x^{-1}\mathrm{e}^{u(n)-u(n-1)}. \tag{17}$$

It is clear that (1) has a discrete invariance under the transformation

$$x \to t, t \to x. \tag{18}$$

Then, after repeating the discussion above, we can get another set of infinitely many truncated symmetries exchanging the spacetime x and t:

$$\sigma \equiv \sigma_i^x(n, h(x)) = \frac{1}{(i-1)!}\sum_{k=0}^{i-1} h^{(i-k-1)}(-\partial_x + \partial_t^{-1}\mathrm{e}^{u(n-1)-u(n)}\Delta(n) - \partial_t^{-1}\mathrm{e}^{u(n)-u(n+1)}\Delta(n+1))^k n^{i-1} \qquad i = 1, 2, 3, \ldots. \tag{19}$$

It is interesting that using the following Lie product definition

$$[A(u), B(u)] = A'B - B'A \equiv \frac{\partial}{\partial\epsilon}[A(u+\epsilon B) - B(u+\epsilon A)]\Big|_{\epsilon=0} \tag{20}$$

each set of symmetries ($\sigma_i^t(n, f)$ or $\sigma_i^x(n, h)$) constitutes a generalized W_∞ type symmetry algebra, say for $\sigma_i^t(n, f) \equiv \sigma_i(f)$,

$$[\sigma_i(f_1), \sigma_j(f_2)] = \sigma_{i+j-2}((j-1)f_1'f_2 - (i-1)f_2'f_1) \qquad i, j \geqslant 1 \qquad \sigma_k(f) = 0 (k < 1). \tag{21}$$

The usual W_∞ algebras [4] which are widely used in various physics fields are just some special subalgebras of (20). For instance, $f = \exp rt$, or $f = t^r$, $r = 0, \pm1, \pm2, \ldots$ in (20). The method used here is easy to extend to the generalized high-dimensional differential-difference systems. The generalized W_∞ symmetry algebras are worthy of further study.

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